練習問題２

1.

（１）$\nabla\varphi=\left(\dfrac{\partial\varphi}{\partial x},\ \dfrac{\partial\varphi}{\partial y},\ \dfrac{\partial\varphi}{\partial z}\right)$

$$=(z^3-2xyz,\quad -x^2z,\quad 3xz^2-x^2y)$$

点$(1,\ 1,\ 1)$における$\nabla\varphi$は,

$$\nabla\varphi=(1-2,\quad -1,\quad 3-1)=(\mathbf{-1},\quad \mathbf{-1},\quad \mathbf{2})$$

（２）点$(1,\ 0,\ 1)$における$\nabla\varphi$は,

$$\nabla\varphi=(1-0,\quad -1,\quad 3-0)=(1,\quad -1,\quad 3)$$

$$|\nabla\varphi|=\sqrt{1^2+(-1)^2+3^2}=\sqrt{11}$$

よって，同じ向きの単位ベクトルは

$$\boldsymbol{\frac{1}{\sqrt{11}}(1,\quad -1,\quad 3)}$$

（３）

$\dfrac{\partial^2\varphi}{\partial x^2}=-2yz,\quad \dfrac{\partial^2\varphi}{\partial y^2}=0,\quad \dfrac{\partial^2\varphi}{\partial z^2}=6xz$であるから,

$$\nabla^2\varphi=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$

$$=-2yz+0+6xz$$

$$=-2yz+6xz$$

点$(1,\ 2,\ 3)$における$\nabla^2\varphi$は,

$$\nabla^2\varphi=-2\cdot2\cdot3+6\cdot1\cdot3$$

$$=-12+18=\mathbf{6}$$

2.

（１）$\nabla\cdot\boldsymbol{a}=\dfrac{\partial}{\partial x}(x^2y)+\dfrac{\partial}{\partial y}(-zx)+\dfrac{\partial}{\partial z}(yz)$

$$=2xy+0+y$$

$$=\boldsymbol{2xy+y}$$

（２）

$$\nabla\times\boldsymbol{a}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & -zx & yz\end{vmatrix}$$

$$=\left\{\frac{\partial}{\partial y}(yz)-\frac{\partial}{\partial z}(-zx)\right\}\boldsymbol{i}-\left\{\frac{\partial}{\partial x}(yz)-\frac{\partial}{\partial z}(x^2y)\right\}\boldsymbol{j}$$

$$+\left\{\frac{\partial}{\partial x}(-zx)-\frac{\partial}{\partial y}(x^2y)\right\}\boldsymbol{k}$$

$$=(z+x)\boldsymbol{i}-(0-0)\boldsymbol{j}+(-z-x^2)\boldsymbol{k}$$

$$=(\boldsymbol{z+x},\quad \mathbf{0},\quad \boldsymbol{-z-x^2})$$

（３）（１）の結果を用いて

$$\nabla(\nabla\cdot\boldsymbol{a})=\left(\frac{\partial}{\partial x}(2xy+y),\quad \frac{\partial}{\partial y}(2xy+y),\quad \frac{\partial}{\partial z}(2xy+y)\right)$$

$$=(\boldsymbol{2y},\quad \boldsymbol{2x+1},\quad \mathbf{0})$$

（４）（２）の結果を用いて

$$\nabla\times(\nabla\times\boldsymbol{a})=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ z+x & 0 & -z-x^2\end{vmatrix}$$

$$=\left\{\frac{\partial}{\partial y}(-z-x^2)-0\right\}\boldsymbol{i}$$

$$-\left\{\frac{\partial}{\partial x}(-z-x^2)-\frac{\partial}{\partial z}(z+x)\right\}\boldsymbol{j}$$

$$+\left\{0-\frac{\partial}{\partial y}(z+x)\right\}\boldsymbol{k}$$

$$=(0-0)\boldsymbol{i}-(-2x-1)\boldsymbol{j}+(0-0)\boldsymbol{k}$$

$$=0\boldsymbol{i}+(2x+1)\boldsymbol{j}+0\boldsymbol{k}$$

$$=(\mathbf{0},\quad \boldsymbol{2x+1},\quad \mathbf{0})$$

3.

（１）

$$\boldsymbol{a}\times\boldsymbol{r}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & 3\\ x & y & z\end{vmatrix}$$

$$=(2z-3y)\boldsymbol{i}-(z-3x)\boldsymbol{j}+(y-2x)\boldsymbol{k}$$

$$=(2z-3y,\quad 3x-z,\quad y-2x)$$

よって,

左辺$=\nabla\cdot(\boldsymbol{a}\times\boldsymbol{r})$

$$=\frac{\partial}{\partial x}(2z-3y)+\frac{\partial}{\partial y}(3x-z)+\frac{\partial}{\partial z}(y-2x)$$

$$=0+0+0$$

$=0=$右辺

（２）$\boldsymbol{a}\cdot\boldsymbol{r}=x+2y+3z$

したがって,

$$(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{b}=(x+2y+3z)(4,\quad 5,\quad 6)$$

$$=(4x+8y+12z,\quad 5x+10y+15z,\quad 6x+12y+18z)$$

よって,

$\nabla \times \big((\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{b}\big)$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4x+8y+12z & 5x+10y+15z & 6x+12y+18z \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(6x+12y+18z) - \frac{\partial}{\partial z}(5x+10y+15z)\right\}\boldsymbol{i}$$

$$-\left\{\frac{\partial}{\partial x}(6x+12y+18z) - \frac{\partial}{\partial z}(4x+8y+12z)\right\}\boldsymbol{j}$$

$$+\left\{\frac{\partial}{\partial x}(5x+10y+15z) - \frac{\partial}{\partial y}(4x+8y+12z)\right\}\boldsymbol{k}$$

$$= (12-15)\boldsymbol{i} - (6-12)\boldsymbol{j} + (5-8)\boldsymbol{k}$$

$$= -3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$$

$$= (-3,\ \ 6,\ \ -3)$$

また，

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= (12-15)\boldsymbol{i} - (6-12)\boldsymbol{j} + (5-8)\boldsymbol{k}$$

$$= -3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$$

$$= (-3,\ \ 6,\ \ -3)$$

よって，$\nabla \times \big((\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{b}\big) = \boldsymbol{a}\times\boldsymbol{b}$

（３）$\boldsymbol{a}\times\boldsymbol{r}$は（１）で求めているので，

$$\boldsymbol{a}\times\boldsymbol{r} = (2z-3y,\ \ 3x-z,\ \ y-2x)$$

よって，

左辺 $= \nabla \times (\boldsymbol{a}\times\boldsymbol{r})$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z-3y & 3x-z & y-2x \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(y-2x) - \frac{\partial}{\partial z}(3x-z)\right\}\boldsymbol{i}$$

$$-\left\{\frac{\partial}{\partial x}(y-2x) - \frac{\partial}{\partial z}(2z-3y)\right\}\boldsymbol{j}$$

$$+\left\{\frac{\partial}{\partial x}(3x-z) - \frac{\partial}{\partial y}(2z-3y)\right\}\boldsymbol{k}$$

$$= (1+1)\boldsymbol{i} - (-2-2)\boldsymbol{j} + (3+3)\boldsymbol{k}$$

$$= 2\boldsymbol{i} + 4\boldsymbol{j} + 6\boldsymbol{k}$$

$$= (2,\ \ 4,\ \ 6)$$

$$= 2(1,\ \ 2,\ \ 3)$$

$= 2\boldsymbol{a}$＝右辺

4.

（１）

$$\nabla\times\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\boldsymbol{k}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z},\ \ \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x},\ \ \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$= \boldsymbol{0}$$

よって，

$$\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} = 0,\quad \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} = 0,\quad \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} = 0$$

したがって，

$$\frac{\partial a_z}{\partial y} = \frac{\partial a_y}{\partial z},\quad \frac{\partial a_x}{\partial z} = \frac{\partial a_z}{\partial x},\quad \frac{\partial a_y}{\partial x} = \frac{\partial a_x}{\partial y}$$

（２）

$$\boldsymbol{a}\times\boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix}$$

$$= (a_y z - a_z y)\boldsymbol{i} - (a_x z - a_z x)\boldsymbol{j} + (a_x y - a_y x)\boldsymbol{j}$$

$$= (\boldsymbol{a_y z - a_z y},\ \ \boldsymbol{a_z x - a_x z},\ \ \boldsymbol{a_x y - a_y x})$$

（３）　（２）の結果より，

右辺 $= \nabla\cdot(\boldsymbol{a}\times\boldsymbol{r})$

$$= \left(\frac{\partial}{\partial x},\ \ \frac{\partial}{\partial y},\ \ \frac{\partial}{\partial z}\right)\cdot(a_y z - a_z y,\ \ a_z x - a_x z,\ \ a_x y - a_y x)$$

$$= \frac{\partial}{\partial x}(a_y z - a_z y) + \frac{\partial}{\partial y}(a_z x - a_x z) + \frac{\partial}{\partial z}(a_x y - a_y x)$$

$$= \frac{\partial a_y}{\partial x}z - \frac{\partial a_z}{\partial x}y + \frac{\partial a_z}{\partial y}x - \frac{\partial a_x}{\partial y}z + \frac{\partial a_x}{\partial z}y - \frac{\partial a_y}{\partial z}x$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)z$$

$= 0 =$ 右辺　　※（１）の結果より

5.

（１）$\boldsymbol{c} = (c_x,\ \ c_y,\ \ c_z)$とする．

$$\boldsymbol{r}\cdot\boldsymbol{c} = (x,\ \ y,\ \ z)\cdot(c_x,\ \ c_y,\ \ c_z)$$

$$= xc_x + yc_y + zc_z$$

左辺 $= \nabla(\boldsymbol{r}\cdot\boldsymbol{c})$

$$= \nabla\left(xc_x + yc_y + zc_z\right)$$

$$= \left(\frac{\partial}{\partial x}(xc_x + yc_y + zc_z),\ \ \frac{\partial}{\partial y}(xc_x + yc_y + zc_z),\ \ \frac{\partial}{\partial z}(xc_x + yc_y + zc_z)\right)$$

$= (c_x,\quad c_y,\quad c_z)$

$= \boldsymbol{c} =$ 右辺

（2）

$\boldsymbol{v} = (\boldsymbol{r}\cdot\boldsymbol{c})\boldsymbol{c}$

$= (xc_x + yc_y + zc_z)(c_x,\quad c_y,\quad c_z)$

$= (xc_x{}^2 + yc_y + zc_z,\quad xc_x + yc_y{}^2 + zc_z,\quad xc_x + yc_y + zc_z{}^2)$

となるから，

$\nabla\cdot\boldsymbol{v} = \left(\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial z}\right)$

$\cdot(xc_x{}^2 + yc_y + zc_z,\quad xc_x + yc_y{}^2 + zc_z,\quad xc_x + yc_y + zc_z{}^2)$

$= c_x{}^2 + c_y{}^2 + c_z{}^2$

また，$|\boldsymbol{c}|^2 = c_x{}^2 + c_y{}^2 + c_z{}^2$

よって，$\nabla\cdot\boldsymbol{v} = |\boldsymbol{c}|^2$

（3）

左辺 $= \nabla\times\boldsymbol{v}$

$= \nabla\times\{(\boldsymbol{r}\cdot\boldsymbol{c})\boldsymbol{c}\}$

$= \nabla(\boldsymbol{r}\cdot\boldsymbol{c})\times\boldsymbol{c} + (\boldsymbol{r}\cdot\boldsymbol{c})(\nabla\times\boldsymbol{c})$　　※（1）より

$= \boldsymbol{c}\times\boldsymbol{c} + (\boldsymbol{r}\cdot\boldsymbol{c})\left\{\left(\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial z}\right)\times(c_x,\quad c_y,\quad c_z)\right\}$

$= \boldsymbol{0} + (\boldsymbol{r}\cdot\boldsymbol{c})\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ c_x & c_y & c_z\end{vmatrix}$

$= (\boldsymbol{r}\cdot\boldsymbol{c})\left\{\left(\frac{\partial c_z}{\partial y} - \frac{\partial c_y}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial c_z}{\partial x} - \frac{\partial c_x}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial c_y}{\partial x} - \frac{\partial c_x}{\partial y}\right)\boldsymbol{k}\right\}$

$= (\boldsymbol{r}\cdot\boldsymbol{c})(0\boldsymbol{i} - 0\boldsymbol{j} + 0\boldsymbol{k})$

$= (\boldsymbol{r}\cdot\boldsymbol{c})(0,\quad 0,\quad 0)$

$= \boldsymbol{0} =$ 右辺

6.

（1）

左辺 $= \nabla\cdot(f(r)\boldsymbol{r})$

$= (\nabla f(r))\cdot\boldsymbol{r} + f(r)(\nabla\cdot\boldsymbol{r})$

$= (f'(r)\nabla r)\cdot\boldsymbol{r} + f(r)(\nabla\cdot\boldsymbol{r})$

$= f'(r)\left(\frac{\boldsymbol{r}}{r}\right)\cdot\boldsymbol{r} + 3f(r)$　p. 27 例題 3 の結果より

$= f'(r)\left(\frac{|\boldsymbol{r}|^2}{r}\right) + 3f(r)$

$= \frac{r^2}{r}f'(r) + 3f(r)$

$= rf'(r) + 3f(r) =$ 右辺

（2）

左辺 $= \nabla\times(f(r)\boldsymbol{r})$

$= (\nabla f(r))\times\boldsymbol{r} + f(r)(\nabla\times\boldsymbol{r})$　p. 27 例題 3 の結果より

$= (f'(r)\nabla r)\times\boldsymbol{r} + f(r)\cdot\boldsymbol{0}$

$= \left(f'(r)\frac{\boldsymbol{r}}{r}\right)\times\boldsymbol{r} + \boldsymbol{0}$

$= \left(\frac{1}{r}f'(r)\right)(\boldsymbol{r}\times\boldsymbol{r})$

$= \left(\frac{1}{r}f'(r)\right)\cdot\boldsymbol{0}$

$= \boldsymbol{0} =$ 右辺